THE TROPICAL j-INVARIANT

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ABSTRACT. If $(Q, \mathcal{A})$ is a marked polygon with one interior point, then a general polynomial $f \in \mathbb{K}[x, y]$ with support $\mathcal{A}$ defines an elliptic curve C_f on the toric surface $X_{\mathcal{A}}$. If $\mathbb{K}$ has a non-archimedean valuation into $\mathbb{R}$ we can tropicalize C_f to get a tropical curve $\mathrm{Trop}(C_f)$. If the Newton subdivision induced by f is a triangulation, then $\mathrm{Trop}(C_f)$ will be a graph of genus one and we show that the lattice length of the cycle of that graph is the negative of the valuation of the j-invariant of C_f.

1. INTRODUCTION

Previous work by Grisha Mikhalkin [13], by Michael Kerber and Hannah Markwig [11] and by Magnus Vigeland [18] shows that the length of the cycle of a tropical curve of genus one has properties which one classically attributes to the j-invariant of an elliptic curve without giving a direct link between these two numbers. In [9] we established such a direct link for plane cubics by showing that the tropicalization of the j-invariant is *in general* the negative of the cycle length. In the present paper we generalize this result to elliptic curves on other toric surfaces using the same methods.

More precisely, if $(Q, \mathcal{A})$ is a marked polygon with one interior point, then a general polynomial $f \in \mathbb{K}[x, y]$ with support $\mathcal{A}$ defines an elliptic curve C_f on the toric surface $X_{\mathcal{A}}$. If $\mathbb{K}$ has a non-archimedean valuation we can tropicalize C_f to get a tropical curve $\mathrm{Trop}(C_f)$. If the Newton subdivision induced by f is a triangulation, then $\mathrm{Trop}(C_f)$ will be a graph of genus one and we show in our main result in Theorem 6.4 that the lattice length of the cycle of the graph is the negative of the valuation of the j-invariant.

In the case where the triangulation is *unimodular*, i.e. all the triangles have area $\frac{1}{2}$, this result was independently derived by David Speyer [17, Proposition 9.2] using Tate uniformization of elliptic curves. David Speyer's result is more general though in the sense that it applies to curves in arbitrary toric varieties.

This paper is organized as follows. In Section 2 we consider toric surfaces defined by a marked lattice polygon with one interior point, we recall the classification of these polygons and we consider the impact on the j-invariant for the corresponding elliptic curves. Section 3 recalls the notion of tropicalization and of plane tropical curves. We then introduce in Section 4 the notion of tropical j-invariant and give a formula to compute it. Section 5 shows that the tropical j-invariant is preserved

by integral unimodular affine transformations. With this preparation we are able to state our main result in Section 6. Section 7 is then devoted to the reduction of the proof to considering only three marked polygons and Section 8 shows how these three cases can be dealt with using procedures from the SINGULAR library `jinvariant.lib` (see [10]) which is available via the URL

$$\text{http://www.mathematik.uni-kl.de/\~{} keilen/en/jinvariant.html.}$$

The actual computations are done using `polymake` [4], TOPCOM [16] and SINGU-LAR [6]. The tropical curves in this paper and their Newton subdivisions were produced using the procedure `drawtropicalcurve` from the SINGULAR library `tropical.lib` (see [8]) which can be obtained via the URL

$$\text{http://www.mathematik.uni-kl.de/\~{} keilen/en/tropical.html.}$$

The authors would like to thank Vladimir Berkovich, Jordan Ellenberg, Bjorn Poonen, David Speyer, Charles Staats, Bernd Sturmfels and John Voigt for valuable discussions.

2. Toric surfaces

Throughout this paper we consider mainly marked polygons $(Q, \mathcal{A})$ such that Q contains a single interior lattice point, where by a *marked polygon* we mean a convex lattice polygon Q in $\mathbb{R}^2$ together with a subset $\mathcal{A} \subseteq Q \cap \mathbb{Z}^2$ of the lattice points of Q containing at least the vertices of Q (cf. [5, Section 2.A]). Fixing a base field $\mathbb{K}$ such a polygon defines a polarized *toric surface*

$$X_{\mathcal{A}} \subset \mathbb{P}_{\mathbb{K}}^{|\mathcal{A}|-1}.$$

In the torus $(\mathbb{K}^*)^2 \subset X_{\mathcal{A}}$ the hyperplane section, say C_f, defined by the linear form $\sum_{(i,j) \in \mathcal{A}} a_{ij} \cdot z_{ij}$ is the vanishing locus of the Laurent polynomial

$$f = \sum_{(i,j) \in \mathcal{A}} a_{ij} \cdot x^i y^j$$

(cf. [5, Chapter 5]). Since the arithmetical genus of the hyperplane sections is the number of interior lattice points of Q (cf. [3, p. 91]), the general hyperplane section will be a smooth *elliptic curve*. The *j-invariant* of such a curve is an element of the base field which characterizes the curve up to isomorphism.

An *integral unimodular affine transformation* of $\mathbb{R}^2$ is an affine map

$$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : \alpha \mapsto A \cdot \alpha + \tau$$

with $\tau \in \mathbb{Z}^2$ and $A \in \mathrm{Gl}_2(\mathbb{Z})$ invertible over the integers. Such an integral uni-modular affine transformation ϕ maps each face of Q to a face of the convex lattice polygon $\phi(Q)$ and preserves thereby the number of lattice points on each face. Moreover, ϕ induces an isomorphism of the polarized toric surfaces $X_{\mathcal{A}}$ and $X_{\phi(\mathcal{A})}$ (cf. [5, Proposition 5.1.2]). From the point of view of toric surfaces it therefore suf-fices to consider the marked polygon $(Q, \mathcal{A})$ only up to integral unimodular affine transformations, and if we suppose $\mathcal{A} = Q \cap \mathbb{Z}^2$ then there are precisely sixteen of them which we divide into two groups, Q_a, Q_b and Q_c respectively $Q_{ca}, \ldots, Q_{cm}$ (see Figure 1, cf. [15] or [14]). We fix the interior point at position $(1, 1)$.
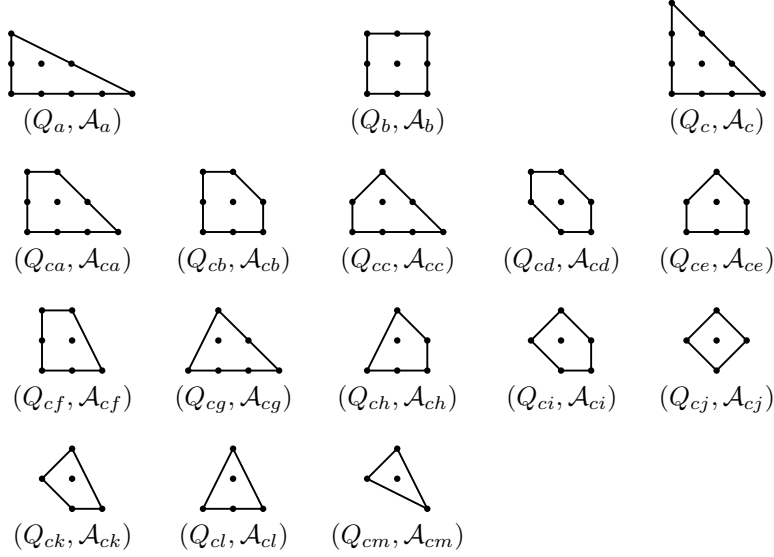
FIGURE 1. The 16 convex lattice polygons with one interior lattice point

The marked polygon $(Q_c, \mathcal{A}_c)$ corresponds to $\mathbb{P}^2_{\mathbb{K}}$ embedded into $\mathbb{P}^9_{\mathbb{K}}$ via the 3-uple Veronese embedding. The marked polygon $(Q_b, \mathcal{A}_b)$ corresponds to $\mathbb{P}^1_{\mathbb{K}} \times \mathbb{P}^1_{\mathbb{K}}$ embedded into $\mathbb{P}^8_{\mathbb{K}}$ via the $(2,2)$-Segre embedding. The marked polygon $(Q_a, \mathcal{A}_a)$ describes the singular weighted projective plane $\mathbb{P}_{\mathbb{K}}(2,1,1)$ embedded into $\mathbb{P}^8$.

If a polygon marked polygon $(Q', \mathcal{A}')$ is derived from $(Q, \mathcal{A})$ by cutting off one lattice point (k, l), like $(Q_{ca}, \mathcal{A}_{ca})$ and $(Q_c, \mathcal{A}_c)$, then the toric surface $X_{\mathcal{A}'}$ is a blow up of $X_{\mathcal{A}}$ in a single point. Moreover, in the torus $(\mathbb{K}^*)^2$ the hyperplane sections corresponding to

$$f = \sum_{(i,j) \in \mathcal{A}'} a_{ij} \cdot x^i y^j = \sum_{(i,j) \in \mathcal{A}} a_{ij} \cdot x^i y^j$$

with $a_{kl} = 0$ coincide. In particular, if they are both smooth their j-invariant coincides since two birationally equivalent curves are already isomorphic (cf. [7, Section I.6]). Since the 13 polygons $Q_{ca}, \ldots, Q_{cm}$ in the second group in Figure 1 are all subpolygons of Q_c the corresponding toric surfaces are all obtained from the projective plane by a couple of blow ups. When we want to compute the j-invariant of the curve corresponding to some Laurent polynomial with support in one of these 13 polygons, we can instead consider the plane curve with support in $\mathcal{A}_c$ but with the appropriate coefficients being zero.

Once we are able to compute the j-invariant for polynomials with support $\mathcal{A}_a$, $\mathcal{A}_b$ and $\mathcal{A}_c$ we are therefore able to compute the j-invariant for every Laurent polynomial with support on the lattice points of a lattice polygon with only one interior point.

We still assume that $(Q, \mathcal{A})$ is a marked lattice polygon with only one interior lattice point as above. Moreover, we use the notation $\underline{a} = (a_{ij} \mid (i,j) \in \mathcal{A})$, and we

suppose that

$$f = \sum_{(i,j)\in\mathcal{A}} a_{ij} \cdot x^i y^j,$$

then the j-invariant

$$j(C_f) = j(f) = \frac{A_{\mathcal{A}}}{B_{\mathcal{A}}}$$

of the curve C_f in $X_{\mathcal{A}}$ defined by f can be expressed as a quotient of two homogeneous polynomials $A_{\mathcal{A}}, B_{\mathcal{A}} \in \mathbb{Q}[\underline{a}]$ of degree 12. In the case of $\mathcal{A} = \mathcal{A}_c$ $A_{\mathcal{A}}$ has 1607 terms and $B_{\mathcal{A}}$ has 2040. In the case of $\mathcal{A} = \mathcal{A}_b$ $A_{\mathcal{A}}$ has 990 terms and $B_{\mathcal{A}}$ has 1010. And finally in the case $\mathcal{A} = \mathcal{A}_a$ $A_{\mathcal{A}}$ has 267 terms and $B_{\mathcal{A}}$ has 312. Every other case can be reduced to these three via some integral unimodular affine transformation and by setting some coefficients equal to zero. The reader interested in seeing or using the polynomials can consult the procedure `invariantsDB` in the Singular library `jinvariant.lib` (see [10]). The proof of our result relies heavily on the investigation of the combinatorics of these polynomials.

3. TROPICALIZATION

In this section we want to pass from the algebraic to the tropical side. For this we specify a field $\mathbb{K}$ with a *non-archimedean valuation* $\mathrm{val} : \mathbb{K}^* \to \mathbb{R}$ as base field and we extend the valuation to $\mathbb{K}$ by $\mathrm{val}(0) = \infty$. We call $\mathrm{val}(k)$ also the *tropicalization* of k. In the examples that we consider $\mathbb{K}$ will always be the *field of Puiseux series*

$$\bigcup_{N=1}^{\infty} \mathrm{Quot}\left(\mathbb{C}\big[\big[t^{\frac{1}{N}}\big]\big]\right) = \left\{ \sum_{\nu=m}^{\infty} c_\nu \cdot t^{\frac{\nu}{N}} \;\middle|\; c_\nu \in \mathbb{C}, N \in \mathbb{Z}_{>0}, m \in \mathbb{Z} \right\}$$

and the valuation of a Puiseux series is its *order*.

If $f = \sum a_{ij} \cdot x^i y^j \in \mathbb{K}[x, y, x^{-1}, y^{-1}]$ is any Laurent polynomial, we call the set

$$\mathrm{supp}(f) = \{(i,j) \in \mathbb{Z}^2 \mid a_{ij} \neq 0\}$$

the *support* of f and the convex hull $\mathrm{N}(f)$ of $\mathrm{supp}(f)$ in $\mathbb{R}^2$ is called the *Newton polygon* of f. If $\mathrm{supp}(f) \subseteq \mathcal{A} \subseteq \mathrm{N}(f) \cap \mathbb{Z}^2$ then f defines a curve C_f in the toric surface $X_{\mathcal{A}}$ as described in Section 2 and we define the *tropicalization* of C_f as

$$\mathrm{Trop}\left(C_f\right) = \overline{\mathrm{val}\left(C_f \cap (\mathbb{K}^*)^2\right)} \subseteq \mathbb{R}^2,$$

i.e. the closure of $\mathrm{val}\left(C_f \cap (\mathbb{K}^*)^2\right)$ with respect to the Euclidean topology in $\mathbb{R}^2$. Here by abuse of notation

$$\mathrm{val} : (\mathbb{K}^*)^2 \longrightarrow \mathbb{Q}^2 : (k_1, k_2) \mapsto \left(\mathrm{val}(k_1), \mathrm{val}(k_2)\right)$$

denotes the Cartesian product of the above valuation map.

A better way to compute the tropicalization of C_f is as the tropical curve defined by the *tropicalization* of the polynomial f, i.e. the piecewise linear map

$$\mathrm{trop}(f) : \mathbb{R}^2 \longrightarrow \mathbb{R} : (x, y) \mapsto \min\{\mathrm{val}(a_{ij}) + i \cdot x + j \cdot y \mid (i,j) \in \mathrm{supp}(f)\}.$$

Given any *plane tropical Laurent polynomial*

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R} : (x, y) \mapsto \min\{u_{ij} + i \cdot x + j \cdot y \mid (i,j) \in \mathcal{A}'\}$$

with *support* $\mathrm{supp}(F) = \mathcal{A}' \subset \mathbb{Z}^2$ finite and $u_{ij} \in \mathbb{R}$, we call the *locus $\mathcal{C}_F$ of non-differentiability* of F, i.e. the set of points $(x, y) \in \mathbb{R}^2$ where the minimum

is attained at least twice, the *plane tropical curve* defined by F. The convex hull $N(F)$ of $\mathrm{supp}(F)$ is again called the *Newton polygon* of F.

By *Kapranov's Theorem* (see [2, Theorem 2.1.1]), $\mathrm{Trop}(C_f)$ coincides with the plane tropical curve defined by the plane tropical polynomial $\mathrm{trop}(f)$. In particular, $\mathrm{Trop}(C_f)$ is a piece-wise linear graph.

The plane tropical Laurent polynomial F induces a *marked subdivision* (cf. [5, Definition 7.2.1]) of the marked polygon $\big(N(F), \mathcal{A}\big)$ with $\mathrm{supp}(F) \subseteq \mathcal{A} \subseteq N(F) \cap \mathbb{Z}^2$ in the following way: project the lower faces of the convex hull of

$$\{(i, j, u_{ij}) \mid (i, j) \in \mathrm{supp}(F)\}$$

into the xy-plane to subdivide of $N(F)$ into smaller polygons and mark those lattice points for which (i, j, u_{ij}) is contained in a lower face.

This subdivision is *dual* to the tropical curve $\mathcal{C}_F$ in the following sense (see [12, Prop. 3.11]): Each marked polygon of the subdivision is dual to a vertex of $\mathcal{C}_F$, and each facet of a marked polygon is dual to an edge of $\mathcal{C}_F$. Moreover, if the facet, say e, has end points (x_1, y_1) and (x_2, y_2) then the *direction vector* $v(E)$ of the dual edge E in $\mathcal{C}_F$ is defined (up to sign) as
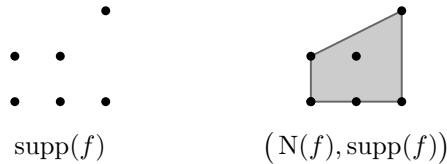
$$v(E) = (y_2 - y_1, x_1 - x_2)^t$$

and points in the direction of E. In particular, the edge E is orthogonal to its dual facet e. Finally, the edge E is unbounded if and only if its dual facet e is contained in a facet of $N(F)$.

Example 3.1
Consider the polynomial

$$f = xy + t \cdot (y + x + x^2 + x^2 y^2) + t^3$$

The following diagram shows the support of f and its marked Newton polygon.

$\mathrm{supp}(f)$ $\big(N(f), \mathrm{supp}(f)\big)$

The tropicalization of f is

$$\mathrm{trop}(f) : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto \min\{x + y, 1 + y, 1 + x, 1 + 2x, 1 + 2x + 2y, 3\}.$$

The support and Newton polygon of f respectively of $\mathrm{trop}(f)$ coincide. In order to compute the marked subdivision of the Newton polygon note that the points

$$(0, 1, 1), (1, 0, 1), (2, 0, 1), (2, 2, 1)$$

lie in a plane while watching from below the point $(1, 1, 0)$ sticks out from this plane and the point $(0, 0, 3)$ lies way above it. We therefore get the following subdivision of the Newton polygon:
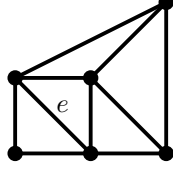
The polygon spanned by $(0,0)$, $(1,0)$ and $(0,1)$ is dual to the vertex of the tropical curve where the terms 3, $1+x$ and $1+y$ take their common minimum, which is at the point $(x,y) = (2,2)$. Similarly the polygon spanned by $(1,0)$, $(1,1)$ and $(0,1)$ corresponds to the point $(x,y) = (1,1)$, and the common face e of the two polygons then is dual to the edge connecting these two points. Note that the direction vector of this edge E is $v(E) = (1,1)$ is orthogonal to the face e connecting the points $(1,0)$ and $(0,1)$ and points from the starting point $(1,1)$ of E to its end point $(2,2)$. Computing the remaining vertices and edges of $\mathrm{Trop}(C_f)$ we get the following graph.
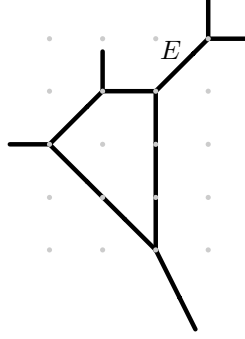
4. The tropical j-invariant of an elliptic plane tropical curve

For the purpose of this paper we want to define an elliptic plane tropical curve in the following way.

Definition 4.1
An *elliptic plane tropical curve* is a tropical curve $\mathcal{C}_F$ defined by a plane tropical Laurent polynomial F whose Newton polygon has precisely one interior lattice point.

The plane tropical curve $\mathcal{C}_F$ in Example 3.1 is elliptic in this sense. Moreover, the graph $\mathcal{C}_F$ has *genus* one, where the genus of a graph is the number of independent cycles of the graph. Obviously a cycle in the graph corresponds to an interior lattice point of the subdivision being a vertex of at least three polygons in the subdivision of the Newton polygon. We want to make this more precise in the following definition.

Definition 4.2
Let $\mathcal{C}$ be a plane tropical curve with marked Newton polygon $(Q, \mathcal{A})$ and with dual marked subdivision $\{(Q_i, \mathcal{A}_i) \mid i = 1, \ldots, l\}$. Suppose that $\tilde{\omega} \in \mathrm{Int}(Q) \cap \mathbb{Z}^2$ and that the $(Q_i, \mathcal{A}_i)$ are ordered such that $\tilde{\omega}$ is a vertex of Q_i for $i = 1, \ldots, k$ and it is not contained in Q_i for $i = k+1, \ldots, l$ (see Figure 2). We then say that $\tilde{\omega}$ *determines a*
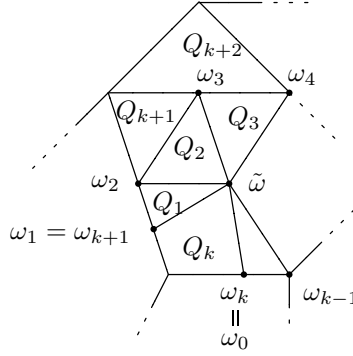
FIGURE 2. Marked subdivision determining a cycle

cycle of $\mathcal{C}$, namely the union of the edges of $\mathcal{C}$ dual to the facets emanating from $\tilde{\omega}$, and we say that these edges *form the cycle* determined by $\tilde{\omega}$. We define the *lattice length of the cycle* to be the sum of the lattice lengths of the edges which form the cycle, where for an edge E with direction vector $v(E)$ (see p. 5) the *lattice length of E* is

$$l(E) = \frac{||E||}{||v(E)||}$$

the Euclidean length of E divided by that of $v(E)$.

Example 4.3
Coming back to our Example 3.1 the curve has one cycle dual to the interior lattice point $(1,1)$ and it consists of four edges $E_1, \ldots, E_4$.
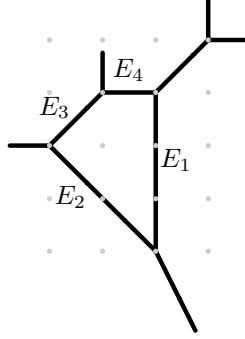
The edge E_1 is dual to the edge e_1 from $(0,1)$ to $(1,1)$ in the Newton subdivision in Example 3.1, so that its direction vector is $v(E_1) = (0,-1)$ of Euclidean length 1 and that the lattice length of E_1 is $l(E_1) = ||E_1|| = 3$. Doing similar computations for the other edges the cycle length is

$$l(E_1) + l(E_2) + l(E_3) + l(E_4) = 3 + 2 + 1 + 1 = 7.$$

Definition 4.4
If $\mathcal{C}$ is an elliptic plane tropical curve then $\mathcal{C}$ has at most one cycle, and we define its *tropical j-invariant* $j_{\mathrm{trop}}(\mathcal{C})$ to be the lattice length of this cycle if it has one. If $\mathcal{C}$ has no cycle we define its tropical j-invariant to be zero.

In Example 4.3 the elliptic plane tropical curve has tropical j-invariant 7.

If a we fix the part of a Newton subdivision which determines the cycle then there is a nice formula to compute the cycle length, and thus the tropical j-invariant. For the proof we refer to [9, Lemma 3.7].

Lemma 4.5
Let $(Q, \mathcal{A})$ be a marked polygon in $\mathbb{R}^2$ with a marked subdivision $\{(Q_i, \mathcal{A}_i) \mid i = 1, \ldots, l\}$ and suppose that $\tilde{\omega} \in \text{Int}(Q) \cap \mathbb{Z}^2$ is a vertex of Q_i for $i = 1, \ldots, k$ and it is not contained in Q_i for $i = k+1, \ldots, l$.

If $u \in \mathbb{R}^{\mathcal{A}}$ is such that the plane tropical curve
$$F = \min\{u_{ij} + i \cdot x + j \cdot y \mid (i,j) \in \mathcal{A}\}$$
induces this subdivision (as described in Section 3), then $\tilde{\omega}$ determines a cycle in the plane tropical curve $\mathcal{C}_F$ and, using the notation in Figure 2, its length is
$$\sum_{j=1}^{k} (u_{\tilde{\omega}} - u_{\omega_j}) \cdot \frac{D_{j-1,j} + D_{j,j+1} + D_{j+1,j-1}}{D_{j-1,j} \cdot D_{j,j+1}}$$
where $D_{i,j} = \det(w_i, w_j)$ with $w_i = \omega_i - \tilde{\omega}$ and $w_j = \omega_j - \tilde{\omega}$.

This formula implies in particular the following corollary.

Corollary 4.6
If $(Q, \mathcal{A})$ is a marked lattice polygon in $\mathbb{R}^2$ with precisely one interior lattice point, then
$$j_{\text{trop}} : \mathbb{R}^{\mathcal{A}} \longrightarrow \mathbb{R} : u \mapsto j_{\text{trop}}(u) := j_{\text{trop}}(\mathcal{C}_{F_u})$$
with
$$F_u = \min\{u_{ij} + i \cdot x + j \cdot y \mid (i,j) \in \mathcal{A}\}$$
is a piecewise linear function which is linear on cones of the secondary fan (cf. [5, Chapter 7]) of $\mathcal{A}$.

5. Unimodular transformations preserve lattice length

We want to relate the classical j-invariant to the tropical j-invariant, and we would again like to reduce the consideration of all possible Newton polygons with one interior point to the 16 polygons in Figure 1, or even better, to the three basic ones in the first group there. For that we have to understand the impact of an integral unimodular affine transformation on a plane tropical Laurent polynomial respectively the induced plane tropical curve.

Given a linear form $l = u + i \cdot x + j \cdot y = u + (i,j) \cdot (x,y)^t$ with $i, j \in \mathbb{Z}$ and $u \in \mathbb{R}$ and given an integral unimodular affine transformation
$$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : \alpha \mapsto A \cdot \alpha + \tau$$
with $A \in \text{Gl}_2(\mathbb{Z})$ and $\tau \in \mathbb{Z}^2$, we let ϕ act on l via
$$l^{\phi} = u + (x,y) \cdot \phi\big((i,j)^t\big)$$
and we let ϕ act on a plane tropical Laurent polynomial $F = \min\{u_{ij} + i \cdot x + j \cdot y \mid (i,j) \in \mathcal{A}'\}$ via the linear forms, i.e.
$$F^{\phi} = \min\{u_{ij} + (x,y) \cdot \phi\big((i,j)^t\big) \mid (i,j) \in \mathcal{A}'\}.$$

Note, that the translation by τ does not change the piecewise linear function defined by F at all and the Newton polygon of F is just translated by τ. So τ has neither any impact on the Newton subdivision of F nor on the tropical curve defined by F. Moreover, it is obvious that if $\{(Q_i, \mathcal{A}_i) \mid i = 1, \ldots, k\}$ is the marked subdivision of $\left(\mathrm{N}(F), \mathrm{supp}(F)\right)$ induced by F, then $\{\phi(Q_i), \phi(\mathcal{A}_i) \mid i = 1, \ldots, k\}$ is the marked subdivision of $\left(\mathrm{N}(F^\phi), \mathrm{supp}(F^\phi)\right)$ induced by F^ϕ.

It is a well-known fact that an integral unimodular affine transformation preserves lattice length, which implies the following corollary.

Corollary 5.1
Let F be a plane tropical Laurent polynomial such that $\mathcal{C}_F$ is elliptic with positive tropical j-invariant and let ϕ be an integral unimodular affine transformation of $\mathbb{R}^2$, then $\mathcal{C}_{F^\phi}$ is elliptic with the same tropical j-invariant
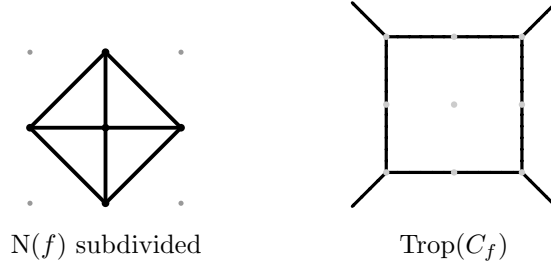
$$j_{\mathrm{trop}}(\mathcal{C}_F) = j_{\mathrm{trop}}(\mathcal{C}_{F^\phi}).$$

Example 5.2
Consider the polynomial

$$f = x^2 y + x y^2 + \frac{1}{t} \cdot xy + x + y$$

inducing the following subdivision of its Newton polygon and the corresponding tropical curve:

N(f) subdivided   Trop(C_f)

The plane tropical curve $\mathrm{Trop}(C_f)$ is elliptic with tropical j-invariant 8. If we now apply the integral unimodular affine transformation

$$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : \alpha \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot \alpha$$

we get

$$f^\phi = x^5 y^3 + x^4 y^3 + \frac{1}{t} \cdot x^3 y^2 + x^2 y + xy$$

with the following subdivision of its Newton polygon and the corresponding elliptic plane tropical curve having again tropical j-invariant 8.

$\mathrm{N}(f^\phi)$ subdivided $\mathrm{Trop}(C_{f^\phi})$
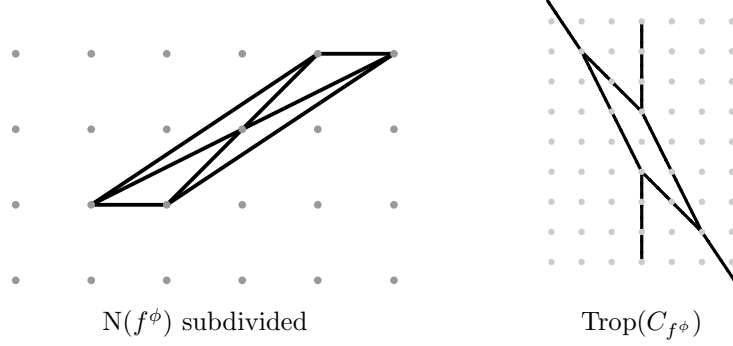
The considerations in Section 2 together with this corollary will allow us to reduce the study of the tropicalization of an elliptic curve in a toric surface with an arbitrary Newton polygon with one interior point to the study of those whose Newton polygons are among the 16 polygons in Figure 1.

6. The main result

Let us suppose now that $(Q, \mathcal{A})$ is a lattice polygon with only one interior point.

Remark 6.1

In Section 2 we have seen that the j-invariant of a curve C_f with $\mathrm{supp}(f) \subseteq \mathcal{A}$ can be computed by plugging the coefficients a_{ij} of f into a suitable quotient $j = \frac{A_\mathcal{A}}{B_\mathcal{A}}$ of homogeneous polynomials $A_\mathcal{A}, B_\mathcal{A} \in \mathbb{Q}[\underline{a}]$. This means in particular, that the valuation of the j-invariant can be read off $A_\mathcal{A}$ and $B_\mathcal{A}$ directly, unless some unlucky cancellation of leading terms occurs.

This leads to the following definition.

Definition 6.2

The *generic valuation* of a polynomial $0 \neq H = \sum_\omega H_\omega \cdot \underline{a}^\omega \in \mathbb{Q}[\underline{a}]$ with $\underline{a} = (a_{ij} \mid (i,j) \in \mathcal{A})$ is

$$\mathrm{val}_H : \mathbb{R}^\mathcal{A} \longrightarrow \mathbb{R} : u \mapsto \mathrm{val}_H(u) = \min\{u \cdot \omega \mid H_\omega \neq 0\},$$

where

$$u \cdot \omega = \sum_{(i,j) \in \mathcal{A}_c} u_{ij} \cdot \omega_{ij}.$$

The *generic valuation of the j-invariant* is the function

$$\mathrm{val}_j : \mathbb{R}^\mathcal{A} \longrightarrow \mathbb{R} : u \mapsto \mathrm{val}_j(u) = \mathrm{val}_{A_\mathcal{A}}(u) - \mathrm{val}_{B_\mathcal{A}}(u).$$

Note that the tropical j-invariant is a *tropical rational function* in the sense of [13, Sec. 2.2] and [1, Def. 3.1].

Remark 6.3

As mentioned above, unless some unlucky cancellation of the leading terms occurs for any $f = \sum_{(i,j) \in \mathcal{A}} a_{ij} \cdot x^i y^j \in \mathbb{K}[x,y]$ with $u_{ij} = \mathrm{val}(a_{ij})$ for all $(i,j) \in \mathcal{A}$ we have

$$\mathrm{val}_j(u) = \mathrm{val}\big(j(f)\big).$$

Note also, that if D is a cone of the Gröbner fan of $A_{\mathcal{A}}$ and D' is a cone of the Gröbner fan of $B_{\mathcal{A}}$ then

$$\mathrm{val}_{j_|} : D \cap D' \longrightarrow \mathbb{R}$$

is *linear* by definition, and if both are top-dimensional, then no unlucky cancellation of leading terms can occur. In particular, the *generic valuation of the j-invariant* val_j is a piece linear function.

We can now state the main result of our paper whose proof is discussed in the subsequent sections.

Theorem 6.4
Let $(Q, \mathcal{A})$ be a lattice polygon with only one interior lattice point.

If $u \in \mathbb{R}^{\mathcal{A}}$ is such that $\mathcal{C}_F$ with

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R} : (x,y) \mapsto \min\{u_{ij} + i \cdot x + j \cdot y \mid (i,j) \in \mathcal{A}\}$$

has a cycle, then

$$\mathrm{val}_j(u) = -j_{\mathrm{trop}}(u).$$

Moreover, if u is in a top-dimensional cone of the secondary fan of $\mathcal{A}$ and $f = \sum_{(i,j)\in\mathcal{A}} a_{ij} \cdot x^i y^j$ with $\mathrm{val}(a_{ij}) = u_{ij}$, then
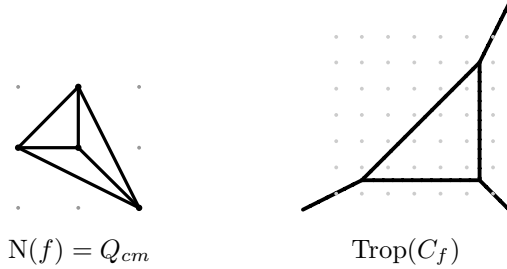
$$\mathrm{val}\left(j(f)\right) = -j_{\mathrm{trop}}(\mathcal{C}_F) = -j_{\mathrm{trop}}\left(\mathrm{Trop}(C_f)\right).$$

Example 6.5
Consider the curve C_f defined by

$$f = t^{\frac{3}{2}} \cdot (y + x^2 + xy^2) + xy$$

with the following subdivision of the Newton polygon and the corresponding elliptic plane tropical curve $\mathrm{Trop}(C_f)$:

$$\mathrm{N}(f) = Q_{cm} \qquad\qquad \mathrm{Trop}(C_f)$$

The vertices of $\mathrm{Trop}(C_f)$ are

$$\left(\frac{3}{2}, 3\right), \quad \left(\frac{3}{2}, -\frac{3}{2}\right) \quad \text{and} \quad \left(-3, -\frac{3}{2}\right),$$

so that its tropical j-invariant is $j_{\mathrm{trop}}\left(\mathrm{Trop}(f)\right) = \frac{27}{2}$, while its j-invariant

$$j(f) = \frac{1 + 72 \cdot t^{\frac{9}{2}} + 1728 \cdot t^9 + 13824 \cdot t^{\frac{27}{2}}}{t^{\frac{27}{2}} + 27 \cdot t^{18}}$$

has valuation $-\frac{27}{2}$.

An immediate consequence of the above theorem is the following corollary.

Corollary 6.6
If $f = \sum_{(i,j)\in\mathcal{A}} a_{ij} \cdot x^i y^j \in \mathbb{K}[x,y]$ defines a smooth elliptic curve in $X_{\mathcal{A}}$ whose j-invariant has non-negative valuation, then $\mathrm{Trop}(C_f)$ has no cycle.

7. REDUCTION TO $\mathcal{A} \in \{\mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c\}$

Using an integral unimodular transformation we may assume that Q is one of the 16 polygons in Figure 1, since the application of such a transformation does not effect the statement of Theorem 6.4 due to Corollary 5.1 and Section 2.

Next we want to reduce to the cases $\mathcal{A} \in \{\mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c\}$.

If $f = \sum_{(i,j)\in\mathcal{A}} a_{ij} \cdot x^i y^j \in \mathbb{K}[x,y]$ with $\mathrm{supp}(f) \subseteq \mathcal{A}$ and we replace f by

$$f' = f + t^{\alpha} \cdot \sum_{(i,j)\in\mathcal{A}\setminus\mathrm{supp}(f)} x^i y^j$$

where α is much larger than $\max\{\mathrm{val}(a_{ij}) \mid (i,j) \in \mathrm{supp}(f)\}$, then obviously

$$\mathrm{val}\big(j(f)\big) = \mathrm{val}\big(j(f')\big).$$

Moreover, if we allow to plug in into val_j points u where some of the u_{ij} are ∞ (as long as the result still is a well defined real number), then we can evaluate val_j at u with $u_{ij} = \mathrm{val}(a_{ij}) \in \mathbb{R} \cup \{\infty\}$, f defines a smooth elliptic curve and we get obviously

$$\mathrm{val}_j(u) = \mathrm{val}_j(u')$$

where $u'_{ij} = u_{ij}$ for $(i,j) \in \mathrm{supp}(f)$ and else $u'_{ij} = \alpha$ with α sufficiently large.

Finally, if in the definition of F_u we allow some u_{ij} to be ∞ then with the above notation the cycle of $\mathcal{C}_{F_u}$ and $\mathcal{C}_{F_{u'}}$ will not change, so that

$$j_{\mathrm{trop}}(u) = j_{\mathrm{trop}}(\mathcal{C}_{F_u}) = j_{\mathrm{trop}}(\mathcal{C}_{F_{u'}}) = j_{\mathrm{trop}}(u').$$

This shows that whenever we may as well assume that $\mathcal{A} \in \{\mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c\}$.

8. THE CASES $\mathcal{A}_a$, $\mathcal{A}_b$ AND $\mathcal{A}_c$

The case $\mathcal{A}_c$ has been treated in [9], and the two other cases work along the same lines. We therefore will be rather short in our presentation. Instead of considering all the cases by hand, as was done in [9] we will refer to computations done using the computer algebra systems `polymake` [4], `TOPCOM` [16] and SINGULAR [6]. The code that we used for this is contained in the SINGULAR library `jinvariant.lib` (see [10]) and it is available via the URL

> http://www.mathematik.uni-kl.de/~ keilen/en/jinvariant.html.

Fix now $(Q,\mathcal{A})$ with $\mathcal{A} \in \{\mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c\}$.

We first of all observe that by Corollary 4.6 the tropical j-invariant is linear on the cones of the secondary fan of $\mathcal{A}$ and that by Lemma 4.5 we can read off the assignment rule on each cone from the Newton subdivision of $(Q,\mathcal{A})$. Moreover, for the statement in Theorem 6.4 we only have to consider such cones for which the interior lattice point of Q is visible in the subdivision.

If $U_{\mathcal{A}} \subseteq \mathbb{R}^{\mathcal{A}}$ is the union of these cones, then it was in each of the cases $\mathcal{A} \in \{\mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c\}$ computed by the procedure `testInteriorInequalities` in the library `jinvariant.lib` that $U_{\mathcal{A}}$ is contained in a single cone of the Gröbner fan of $A_{\mathcal{A}}$ and that

$$\mathrm{val}_{A_{\mathcal{A}}|} : U_{\mathcal{A}} \longrightarrow \mathbb{R} : u \mapsto 12 \cdot u_{11}.$$

It suffices therefore to show that $\mathrm{val}_{B_{\mathcal{A}}}$ is linear on the cones of the secondary fan of $\mathcal{A}$ and to compare the assignment rules for val_j and j_{trop} on each of these cones.

The two marked polygons $(Q_b, \mathcal{A}_b)$ and $(Q_c, \mathcal{A}_c)$ define *smooth* toric surfaces and in these cases $B_{\mathcal{A}} = \Delta_{\mathcal{A}}$ is the $\mathcal{A}$-discriminant of $\mathcal{A}$ (cf. [5, Chapter 9]). Therefore, by the Prime Factorization Theorem (see [5, Theorem 10.1.2]) the secondary fan of $\mathcal{A}_b$ respectively $\mathcal{A}_c$ is a refinement of the Gröbner fan of the $B_{\mathcal{A}_b}$ respectively $B_{\mathcal{A}_c}$. In view of Remark 6.3 and by the above considerations this shows in particular that val_j is linear on each cone of the secondary fan of $\mathcal{A}$ for $\mathcal{A} \in \{\mathcal{A}_b, \mathcal{A}_c\}$ which is contained in $U_{\mathcal{A}}$. The comparison of the assignment rules of the two linear functions val_j and j_{trop} on each of the cones contained in $U_{\mathcal{A}}$ was done by the procedure `displayFan` from the SINGULAR library `jinvariant.lib` using TOPCOM and `polymake`. It produces two postscript files which show all the different cases together with the assignment rules. The files are available via

http://www.mathematik.uni-kl.de/˜ keilen/download/Tropical/secondary_fan_of_2x2.ps

for $\mathcal{A} = \mathcal{A}_b$ respectively via

http://www.mathematik.uni-kl.de/˜
keilen/download/Tropical/secondary_fan_of_cubic.ps

for $\mathcal{A} = \mathcal{A}_c$ respectively. 849 cases have to be considered for $\mathcal{A} = \mathcal{A}_c$ and 255 for $\mathcal{A} = \mathcal{A}_b$.

In the case $\mathcal{A} = \mathcal{A}_a$ the toric surface $X_{\mathcal{A}}$ is *not smooth*, but a quadric cone. Moreover, in this case $B_{\mathcal{A}}$ is *not* the $\mathcal{A}$-discriminant $\Delta_{\mathcal{A}}$, but instead

$$B_{\mathcal{A}} = u_{02}^2 \cdot \Delta_{\mathcal{A}}.$$

Thus, the Gröbner fan of $B_{\mathcal{A}}$ coincides with the Gröbner fan of $\Delta_{\mathcal{A}}$ and it is still true by the Prime Factorization Theorem that the secondary fan of $\mathcal{A}$ is a refinement of the Gröbner fan of $B_{\mathcal{A}}$. We can therefore argue as above, and the case distinction (202 cases) can be viewed via

http://www.mathematik.uni-kl.de/˜ keilen/download/Tropical/secondary_fan_of_4x2.ps.

This finishes our proof, where for the "moreover" part we take Remark 6.3 into account.

Remark 8.1
It follows from the proof that j_{trop} is indeed linear on each cone of the Gröbner fan of $B_{\mathcal{A}}$ in the above cases. This could have been proved directly with the same argument as in [9, Lemma 5.2].

If we denote by $D_{\mathcal{A}}$ the regular $\mathcal{A}$-determinant (cf. [5, Section 11.1]), then $D_{\mathcal{A}_b} = \Delta_{\mathcal{A}_b}$ and $D_{\mathcal{A}_c} = \Delta_{\mathcal{A}_c}$ by [5, Theorem 11.1.3] since $X_{\mathcal{A}}$ is smooth in these cases. Even though in general the regular $\mathcal{A}$-determinant is not a polynomial, it is so for

$\mathcal{A} = \mathcal{A}_a$ by [5, Theorem 11.1.6] since $\mathcal{A}_a$ is quasi-smooth by [5, Theorem 5.4.12] in the sense of that theorem. More precisely, we have

$$D_{\mathcal{A}_a} = u_{02} \cdot \Delta_{\mathcal{A}_a}$$

and by [5, Theorem 11.1.3] it is a divisor of the principal $\mathcal{A}$-determinant $E_{\mathcal{A}}$ (cf. [5, Chapter 9]). Therefore, the secondary fan of $\mathcal{A}$ (which is the Gröbner fan of $E_{\mathcal{A}}$) is a refinement of the Gröbner fan of $D_{\mathcal{A}}$ and thus of the Gröbner fan of $B_{\mathcal{A}}$.

One therefore could have used the description of the vertices of the Newton polytope of $D_{\mathcal{A}}$ in [5, Theorem 11.3.2] in order to show that on each cone of the Gröbner fan of $B_{\mathcal{A}}$ contained in $U_{\mathcal{A}}$ the two functions val_j and j_{trop} coincide by a direct case study as was done in [9, Lemma 5.5] for the case $\mathcal{A} = \mathcal{A}_c$.

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